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Classical integrable field theories in discrete (2 + 1)-dimensional spacetime

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Abstract
We study the 'circular net' (discrete orthogonal net) equations for the angular data generalized by external spectral parameters. A criterion defining physical regimes of these Hamiltonian equations is the reality of the Lagrangian density. There are four distinct regimes for fields and spectral parameters classified by four types of spherical or hyperbolic triangles. Nonzero external spectral parameters provide the existence of field-theoretical ground states and soliton excitations. Spectral parameters of a spherical triangle correspond to a statistical mechanics; spectral parameters of hyperbolic triangles correspond to three different field theories with massless anisotropic dispersion relations.

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Introduction

The 'circular' or 'conic net' (or discrete orthogonal net) equations for angular data [4, 5, 8] take a selected place among all the classical integrable systems [3] on the cubic lattice with an AKP-type hierarchy. Algebraically, these equations arise as a Hamiltonian form of a discrete three-wave system [6, 20]. The 'conic net' equations are the classical $q \to 1$ limit of the quantum 'q-oscillator' model [2]—the top of a pyramid of three-dimensional quantum models—which guarantees in classics the existence of Lagrangian density, energy/action and variational principle [1]. The existence of the quantum counterpart is an evident advantage of discrete spacetime models with respect to their continuous spacetime predecessors [22, 23]. A straightforward geometrical condition for the conic net equations is the reality of angular dynamical variables [1] of a circular net in the Euclidean target space or of an orthochronous hyperbolic net in Minkowski one[1]. However, these discrete differential geometric

[1] The Euclidean sphere of Miquel's theorem corresponds to a one-sheet hyperboloid in the Minkowski metric.

conditions can be essentially extended by a 'physical' condition of the reality of action near the equilibrium point.

A general complex solution of any AKP-type system in finite volume or a solution of the Cauchy problem with generic initial data involves algebraic geometry [15]. Such a general solution of the discrete 'generalized conic net' equations in finite volume has been known for a long time [9–14]. It involves a flat algebraic spectral curve Γ_g of genus $g \leqslant (N-1)^2$ for a size $N^{\times 3}$ cubic lattice (three-periodic boundary conditions), Θ-functions on Jacobian of Γ_q and spectral parameters—three meromorphic functions on Γ_g. A reduction of Γ_g to a sphere gives a g-soliton solution (plane wave solitons) [17, 18]. The soliton regime is the field-theoretical one since solitons are continuous excitations over a ground state—the zero soliton homogeneous solution of the equations of motion. Spectral parameters in the soliton regime are a triple of complex numbers, they enter directly into the equations of motion as extra parameters providing the existence of the homogeneous solution. The spectral parameters break the straightforward discrete-geometric interpretation of the equations of motion.

These two principles—the reality of energy/action and the existence of a homogeneous solution for 'nonzero' spectral parameters—are the starting points for the classification of physical field theories and statistical mechanics for 'generalized conic net' equations. There are four distinct regimes of spectral parameters and corresponding regimes of dynamic fields providing the reality of action. Parameterizations of ground states have a structure of cosine theorems for spherical or various hyperbolic triangles. In the case of a spherical triangle, the ground state is the absolute minimum of energy functional and thus it corresponds to the statistical mechanics. In three-field theoretical cases of hyperbolic triangles a solution of the equations of motion provides an extremum of action; a value of whole action on the soliton solution does not depend on amplitudes of solitons. Expressions for soliton waves involve projective coordinates of hyperbolic triangles; a general field-theoretical solution of equations of motion is a set of soliton–antisoliton pairs analogues to elementary stationary waves. In the low energy–momentum limit the soliton plane waves have a cone-type (anisotropic massless) dispersion relation.

All the regimes of spectral parameters and dynamical fields have manifested counterparts in quantum case. Regimes of fields correspond to classical limits of different representations of q-oscillator algebra; related regimes of spectral parameters correspond to either real or unitary quantum R-matrices. However, relations between quantum theories and classical theories are not straightforward.

This paper is organized as follows. In section 1, we fix notations for the 'conic net' equations generalized by the spectral parameters. Following [1], Lagrangian density is defined in section 2. Next, in section 3 we classify ground states. Soliton solutions of the equations of motion and dispersion relations are defined in section 4. In section 5, we describe the quantum counterparts of our four regimes. Finally, in section 6, we discuss roughly a place of finite gap solutions.

1. Generalized conic net equations

Let $\boldsymbol{n}$ be a node of a large simple cubic lattice

$$\boldsymbol{n} = (n_1, n_2, n_3), \qquad n_i \in \mathbb{Z}. \tag{1}$$

Let $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ be the unit vectors for the lattice,

$$\boldsymbol{e}_1 = (1,0,0), \qquad \boldsymbol{e}_2 = (0,1,0), \qquad \boldsymbol{e}_3 = (0,0,1), \tag{2}$$

so that $\boldsymbol{n} = n_1\boldsymbol{e}_1 + n_2\boldsymbol{e}_2 + n_3\boldsymbol{e}_3$. With each $(\boldsymbol{n} - \boldsymbol{e}_i, \boldsymbol{n})$ edge of the cubic lattice we associate a doublet of dynamical variables

$$\mathcal{A}_{i,\boldsymbol{n}} = \left(k_{i,\boldsymbol{n}}, a^{\pm}_{i,\boldsymbol{n}}\right), \qquad \text{such that} \quad k^2_{i,\boldsymbol{n}} \stackrel{\text{def}}{=} 1 - a^+_{i,\boldsymbol{n}} a^-_{i,\boldsymbol{n}}. \tag{3}$$

The last relation here is in fact the 'conic net' condition. The local equations of motion relate the neighbors of every node $\boldsymbol{n}$,

$$\mathcal{A}_i = \mathcal{A}_{i,\boldsymbol{n}} \quad \text{on} \quad (\boldsymbol{n} - \boldsymbol{e}_i, \boldsymbol{n}) \qquad \text{and} \qquad \mathcal{A}'_i = \mathcal{A}_{i,\boldsymbol{n}+\boldsymbol{e}_i} \quad \text{on} \quad (\boldsymbol{n}, \boldsymbol{n} + \boldsymbol{e}_i), \tag{4}$$

as follows:

$$\begin{aligned}
\left(k_2 a^{\pm}_1\right)' &= u^{\pm 1}_1\left(k_3 a^{\pm}_1 + u^{\mp 1}_2 k_1 a^{\pm}_2 a^{\mp}_3\right),\\
\left(a^{\pm}_2\right)' &= a^{\pm}_1 a^{\pm}_3 - u^{\mp 1}_2 k_1 k_3 a^{\pm}_2,\\
\left(k_2 a^{\pm}_3\right)' &= u^{\pm 1}_3\left(k_1 a^{\pm}_3 + u^{\mp 1}_2 k_3 a^{\mp}_1 a^{\pm}_2\right),
\end{aligned} \tag{5}$$

and

$$k_1 k_2 = k'_1 k'_2, \qquad k_2 k_3 = k'_2 k'_3. \tag{6}$$

$\mathbb{C}$-valued parameters u_i—exponents of spectral parameters—are the same for all nodes $\boldsymbol{n}$.

This classical system can be viewed as an extension of the discrete three-wave equations and conic nets since the last ones correspond to trivial spectral parameters. The circular net in the Euclidean geometry is described by the regime $k^2 > 0$, real $a^{\pm}$ and $u_1 = -u_2 = u_3 = 1$. The Euclidean 'circular net' point has a smooth continuous limit—the classical three-wave resonant equations [23]. The hyperbolic net in the Minkowski geometry is described by the regime $k^2 < 0$, real $a^{\pm}$ and $u_1 = u_2 = u_3 = 1$. For non-trivial spectral parameters or for complex $a^{\pm}$, a geometrical interpretation of equations (5) is unclear.

However, relaxing the geometric condition, equations (5) defines an evolution in $(2 + 1)$-dimensional spacetime. Discrete time is $t = n_1 + n_2 + n_3$, so that (5) literally gives the map from time t to time $t + 1$. A straightforward way to introduce space-like coordinates is to take $n_1 = x$ and $n_3 = y$ so that

$$\boldsymbol{n} = \underbrace{n_1(\boldsymbol{e}_1 - \boldsymbol{e}_2)}_{x\boldsymbol{e}_x} + \underbrace{n_3(\boldsymbol{e}_3 - \boldsymbol{e}_2)}_{y\boldsymbol{e}_y} + \underbrace{(n_1 + n_2 + n_3)\boldsymbol{e}_2}_{t\boldsymbol{e}_t}. \tag{7}$$

The Cauchy problem is well posed for a finite size space-like surface,

$$x, y \in \mathbb{Z}_N, \qquad N \gg 1, \qquad t \in \mathbb{Z}_{\geqslant 0}. \tag{8}$$

The evolution corresponds to a relativistic field theory since the locality of the evolution map provides the relativistic casuality. Note, there is no way to introduce a usual local Hamiltonian for classical discrete time evolution; for instance, a principal object of corresponding quantum theories is a discrete time Heisenberg evolution operator. The framework of statistical mechanics implies 3D Dirichlet or 3D periodical boundary conditions for system (5),

$$n_i \in \mathbb{Z}_N, \qquad i = 1, 2, 3, \qquad N \gg 1. \tag{9}$$

The principal difference between a statistical mechanics and a field theory is that for the given reality regime and 3D periodical boundary conditions a statistical mechanics has a unique ground-state minimizing energy, while a field theory with a saddle-type action has a rich structure of stationary modes.

Equations (5) is a canonical transformation preserving locally the q-oscillator symplectic form [2],

$$\sum_{i=1}^{3} \frac{da^+_i \wedge da^-_i}{k^2_i} = \sum_{i=1}^{3} \frac{da'^+_i \wedge da'^-_i}{k'^2_i}, \tag{10}$$

and thus they have the discrete-type Hamiltonian structure [1]:

$$\log|k_i'| = \frac{1}{2}v_i'\frac{\partial}{\partial v_i'}\widetilde{G}(v;v'), \qquad \log|k_i| = -\frac{1}{2}v_i\frac{\partial}{\partial v_i}\widetilde{G}(v;v'), \tag{11}$$

where

$$v_i \stackrel{\text{def}}{=} \frac{a_i^+}{a_i^-} \tag{12}$$

is a useful canonical partner to k_i, the variables v_j, v_j', k_j, k_j' are related by (5) and $\widetilde{G}(v;v')$ is a generating function of map (5).

It is more convenient to treat k, k' related by (6) as the independent variables and define $G(k;k')$ by

$$dG(k;k') = \frac{1}{2}\sum_{i=1}^{3}(\log[v_i']\,\mathrm{d}\log k_i' - \log[v_i]\,\mathrm{d}\log k_i). \tag{13}$$

Here

$$[v] = v \quad \text{if } v \text{ is positive or unitary and} \quad [v] = -v \quad \text{if } v \text{ is real negative}. \tag{14}$$

Negative v corresponds to the regime $k^2 > 1$. Functions $G(k;k')$ and $\widetilde{G}(v;v')$ are related to the Legendre transform,

$$\widetilde{G}(v;v') = \frac{1}{2}\left(\sum_{i=1}^{3}\log[v_i']\log|k_i'| - \log[v_i]\log|k_i|\right) - G(k;k'). \tag{15}$$

Function $G(k;k')$ is preferable since k_i are the field coordinate-type variables with a fixed reality regime $k^2 \in \mathbb{R}$.

The sum of the local generating functions over all nodes,

$$\mathscr{A} \sim \sum_{\boldsymbol{n}\in\mathbb{Z}^3}\widetilde{G}(v_{i,\boldsymbol{n}}; v_{i,\boldsymbol{n}+\boldsymbol{e}_i}), \tag{16}$$

gives an action/energy for the whole lattice. The equations of motion are the extremum conditions for (16) [1].

Considering the equations of motion as the quasi-classical limit of the quantum models, we expect two distinct regimes: $k^2 < 0$ for modular representation of q-oscillators and $k^2 > 0$ for Fock space representations (regime of unitary k for cyclic representations we do not consider here). Also, it is evident from (13) that $\mathscr{A}$ has two regimes of reality: either the regime of real v when G is manifestly real or the regime of unitary v when $\mathrm{i}G$ is real. Thus we expect *a priori* the existence of four distinct regimes:

$$\begin{array}{llllll} \text{regime 1:} & k_i^2 < 0, & |v_i| = 1, & \text{regime 2:} & k_i^2 < 0, & v_i \in \mathbb{R}, \\ \text{regime 3:} & k_i^2 > 0, & |v_i| = 1, & \text{regime 4:} & k_i^2 > 0, & v_i \in \mathbb{R}. \end{array} \tag{17}$$

The presence of the external parameters u_i in the equations of motion is of great importance for the classification scheme because the regime $v_i \in \mathbb{R}$ corresponds to $u_i \in \mathbb{R}$ and the regime $|v_i| = 1$ corresponds to $|u_i| = 1$.

Also note a signs symmetry of equations (5): the change of the sign of any *two* parameters of u_i followed by a change of signs of corresponding $a_{i,\boldsymbol{n}}^{\pm}$ on, e.g. even edges (n_i in $\boldsymbol{n} = (n_1, n_2, n_3)$ – even) does not change the whole set of the equations of motion. This transformation preserves the variables $v_{i,\boldsymbol{n}}$. In contrast to this, a change of a sign of a single u_i changes the structure of the solution of the equations of motion substantially.

2. Generating function

The (v, k) form of the local equations of motion (5) is the following [1]. Let

$$\begin{aligned}
&\tau_0 = \frac{v_2'}{v_1' v_3'} u_1^2 u_3^2, \qquad \tau_1 = \frac{v_2}{v_1' v_3} \frac{u_1^2}{u_2^2}, \qquad \tau_2 = \frac{v_2'}{v_1 v_3}, \qquad \tau_3 = \frac{v_2}{v_1 v_3'} \frac{u_3^2}{u_2^2}, \\
&\tau_0' = \frac{v_2}{v_1 v_3} \frac{1}{u_2^2}, \qquad \tau_1' = \frac{v_2'}{v_1 v_3'} u_3^2, \qquad \tau_2' = \frac{v_2}{v_1' v_3'} \frac{u_1^2 u_3^2}{u_2^2}, \qquad \tau_3' = \frac{v_2'}{v_1' v_3} u_1^2,
\end{aligned} \tag{18}$$

so that $\tau_0\tau_0' = \tau_i\tau_i'$ and

$$\tau_i\tau_j = \tau_k'\tau_l', \qquad i, j, k, l = \text{any permutation of} \quad (0, 1, 2, 3). \tag{19}$$

Equations (5) provide

$$\begin{aligned}
k_1^2 &= \frac{(1-\tau_2)(1-\tau_3)}{(1-\tau_0')(1-\tau_1')}, \qquad & k_1'^2 &= \frac{(1-\tau_2')(1-\tau_3')}{(1-\tau_0)(1-\tau_1)} \\
k_2^2 &= \frac{(1-\tau_0')(1-\tau_2')}{(1-\tau_1)(1-\tau_3)}, \qquad & k_2'^2 &= \frac{(1-\tau_0)(1-\tau_2)}{(1-\tau_1')(1-\tau_3')}, \\
k_3^2 &= \frac{(1-\tau_1)(1-\tau_2)}{(1-\tau_0')(1-\tau_3')}, \qquad & k_3'^2 &= \frac{(1-\tau_1')(1-\tau_2')}{(1-\tau_0)(1-\tau_3)}.
\end{aligned} \tag{20}$$

Note that the expressions for $k_i^2, k_i'^2$ are invariant with respect to inversion $\tau_j \to \tau_j^{-1}, \tau_j' \to \tau_j'^{-1}$. Thus, differential (13) has the form

$$\begin{aligned}
\mathrm{d}G(k; k') = {}& \frac{1}{4}\sum_{j=0}^{3}(\log[\tau_j]\,\mathrm{d}\log(1-\tau_j) - \log[\tau_j']\,\mathrm{d}\log(1-\tau_j')) \\
&+ \frac{1}{2}\big(\log u_1^2\,\mathrm{d}\log k_1' - \log u_2^2\,\mathrm{d}\log k_2 + \log u_3^2\,\mathrm{d}\log k_3'\big).
\end{aligned} \tag{21}$$

Integrating this, one gets

$$G(k; k') = G_0(k; k') + \frac{1}{2}\big(\log u_1^2 \log|k_1'| - \log u_2^2 \log|k_2| + \log u_3^2 \log|k_3'|\big), \tag{22}$$

where

$$G_0(k; k') = \frac{1}{4}\sum_{j=0}^{3}(J(\tau_j) - J(\tau_j')) \tag{23}$$

and

$$J(\tau) = \begin{cases} \displaystyle\int_{z_0}^{\tau} \log z\,\mathrm{d}\log(1-z) & \text{in regimes } 1, 2 \\ \displaystyle\int_{z_0}^{-\tau} \log z\,\mathrm{d}\log(1+z) & \text{in regimes } 3, 4. \end{cases} \tag{24}$$

The choice of z_0 common for all integrals is irrelevant. We choose zero value for a constant of integration in (22). The $G_0(k; k')$ term in (22) corresponds to Lagrangian density without fields [1] but it must be treated carefully due to the inversion symmetry of τ_j.

The generation function (22) becomes symmetric after an elementary gauge transformation,

$$G_{\text{sym}}(k; k') = G_0(k; k') + \tfrac{1}{4}\big(\log u_1^2 \log|k_1k_1'| - \log u_2^2 \log|k_2k_2'| + \log u_3^2 \log|k_3k_3'|\big). \tag{25}$$

3. Ground states

By 'ground state' we understand the *homogeneous* solutions of equations (5):

$$v_j = v'_j, \qquad k_j = k'_j, \tag{26}$$

which correspond to

$$\frac{v_2}{v_1 v_3} = \frac{u_2}{u_1 u_3} \quad \Leftrightarrow \quad \tau'_j = \tau_j^{-1}. \tag{27}$$

In the vicinity of this point the fields are parameterized by

$$\begin{aligned}
k_1^2 &= \frac{(u_1u_3-u_2)(u_1u_2-u_3)}{(1-u_1u_2u_3)(u_2u_3-u_1)}\,\mathrm{e}^{2\rho_1}, & k_1'^2 &= \frac{(u_1u_3-u_2)(u_1u_2-u_3)}{(1-u_1u_2u_3)(u_2u_3-u_1)}\,\mathrm{e}^{2\rho'_1},\\
k_2^2 &= \frac{(1-u_1u_2u_3)(u_1u_3-u_2)}{(u_2u_3-u_1)(u_1u_2-u_3)}\,\mathrm{e}^{2\rho_2}, & k_2'^2 &= \frac{(1-u_1u_2u_3)(u_1u_3-u_2)}{(u_2u_3-u_1)(u_1u_2-u_3)}\,\mathrm{e}^{2\rho'_2},\\
k_3^2 &= \frac{(u_2u_3-u_1)(u_1u_3-u_2)}{(1-u_1u_2u_3)(u_1u_2-u_3)}\,\mathrm{e}^{2\rho_3}, & k_3'^2 &= \frac{(u_2u_3-u_1)(u_1u_3-u_2)}{(1-u_1u_2u_3)(u_1u_2-u_3)}\,\mathrm{e}^{2\rho'_3},
\end{aligned} \tag{28}$$

where

$$\rho_1 - \rho'_1 = \rho'_2 - \rho_2 = \rho_3 - \rho'_3. \tag{29}$$

The generating function (25) near the ground state $\rho = 0$ is

$$\begin{aligned}
G_{\mathrm{sym}} = G_{\mathrm{sym}}|_{\rho=\rho'=0} &+ \sum_i \frac{\left(u_i - u_i^{-1}\right)}{\left(u_j - u_j^{-1}\right)\left(u_k - u_k^{-1}\right)} x_i^2\\
&- \frac{u_1+u_1^{-1}}{u_1-u_1^{-1}} x_2x_3 + \frac{u_2+u_2^{-1}}{u_2-u_2^{-1}} x_1x_3 - \frac{u_3+u_3^{-1}}{u_3-u_3^{-1}} x_1x_2\\
&- \frac{(u_1u_2u_3-1)(u_1u_2-u_3)(u_1u_3-u_2)(u_2u_3-u_1)}{u_1^2u_2^2u_3^2\left(u_1-u_1^{-1}\right)\left(u_2-u_2^{-1}\right)\left(u_3-u_3^{-1}\right)}\delta^2 + \mathcal{O}(\rho^3),
\end{aligned} \tag{30}$$

where

$$x_i = \frac{\rho_i + \rho'_i}{2}, \qquad \delta^2 = \left(\frac{\rho_i - \rho'_i}{2}\right)^2, \tag{31}$$

and $G_{\mathrm{sym}}|_{\rho=\rho'=0} = G(k;k)$ is calculated at the point $\rho_i = \rho'_i = 0$ ($k_i = k'_i$).

Now we are ready to classify the generating functions for all the four regimes. Depending on the regime, expressions (28) are equivalent to cosine theorems for spherical or hyperbolic triangles; thus the classification scheme is based on the spherical and hyperbolic geometry.

3.1. Regime 1

Unitary spectral parameters are given by

$$u_1 = \mathrm{e}^{\mathrm{i}\epsilon_1\phi_1}, \qquad u_2 = \mathrm{e}^{\mathrm{i}\epsilon_2\phi_2}, \qquad u_3 = \mathrm{e}^{\mathrm{i}\epsilon_3\phi_3}, \tag{32}$$

where $\phi_i > 0$ and ϵ_i are signs. In what follows we use short notations

$$\beta_4 = \frac{\phi_1+\phi_2+\phi_3}{2}, \qquad \beta_i = \beta_4 - \phi_i. \tag{33}$$

In spherical geometry the angles ϕ_i are the sides of a spherical triangle. It is convenient to define excess β_0 instead of half-perimeter β_4 by

$$\beta_0 = \pi - \beta_4. \tag{34}$$

Then for $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$ one has

$$k_1^2 = -\tan^2\frac{\theta_1}{2}, \qquad k_2^2 = -\cot^2\frac{\theta_2}{2}, \qquad k_3^2 = -\tan^2\frac{\theta_3}{2}, \tag{35}$$

where θ_i are the dihedral angles of the spherical triangle with sides ϕ_i:

$$\cos\theta_i = \frac{\cos\phi_i - \cos\phi_j\cos\phi_k}{\sin\phi_j\sin\phi_k}, \qquad \cos\phi_i = \frac{\cos\theta_i + \cos\theta_j\cos\theta_k}{\sin\theta_j\sin\theta_k},$$
$$\tan^2\frac{\theta_i}{2} = \frac{\sin\beta_j\sin\beta_k}{\sin\beta_0\sin\beta_i}. \tag{36}$$

Spherical geometry implies positive θ excess

$$\theta_1+\theta_2+\theta_3 > \pi \quad \Rightarrow \quad 0 < \beta_j < \pi, \qquad j = 0, 1, 2, 3. \tag{37}$$

Other choices of signs ϵ_i such that $\epsilon_1\epsilon_2\epsilon_3 = 1$ are equivalent to crossing transformations of the spherical triangle (in fact, crossing transformations involve $\phi \to \pi - \phi$, the sign symmetry of u_i we discussed above).

For the arbitrary signs ϵ_i let

$$F(k;k') = \mathrm{i}\epsilon_1\epsilon_2\epsilon_3 G(k;k'), \qquad \mathscr{H} = \sum_{\boldsymbol{n}\in\mathbb{Z}^3} F(k_{i,\boldsymbol{n}}; k_{i,\boldsymbol{n}+\boldsymbol{e}_i}). \tag{38}$$

The homogeneous solution provides the absolute minimum of functional $\mathscr{H}$; on this ground state the free-energy density $F(k;k) = F_0$ is given by

$$F_0 = \sum_{j=0}^{3} Л(\beta_j) = \sum_{i=1}^{3} Л(\beta_i) - Л(\beta_4) > 0, \tag{39}$$

where Milnor's Lobachevski function is

$$Л(\beta) = -\int_0^\beta \log(2\sin x)\,\mathrm{d}x. \tag{40}$$

The statement about the absolute minimum can be verified instantly in the free-field approximation (30) where

$$\mathscr{H} = N^3F_0 + \text{positively defined quadratic form of } \rho_{i,\boldsymbol{n}}. \tag{41}$$

Here N^3 is a volume of the lattice. We will discuss this statement beyond the free-field approximation in the following section.

3.2. Regime 2

This is the case of spectral parameters

$$u_1 = \mathrm{e}^{\epsilon_1\phi_1}, \qquad u_2 = \mathrm{e}^{\epsilon_2\phi_2}, \qquad u_3 = \mathrm{e}^{\epsilon_3\phi_3}, \tag{42}$$

where $\phi_i > 0$ and ϵ_i are signs. Values of k_i of the homogeneous solution for $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$ are given by (35) where θ_i are the dihedral angles of a triangle on the upper sheet of a two-sheets hyperboloid. Parameters ϕ_i are the hyperbolic sides of this triangle. The cosine theorems read

$$\cos\theta_i = \frac{\cosh\phi_j\cosh\phi_k - \cosh\phi_i}{\sinh\phi_j\sinh\phi_k}, \qquad \cosh\phi_i = \frac{\cos\theta_i + \cos\theta_j\cos\theta_k}{\sin\theta_j\sin\theta_k},$$
$$\tan^2\frac{\theta_i}{2} = \frac{\sinh\beta_j\sinh\beta_k}{\sinh\beta_4\sinh\beta_i}, \tag{43}$$

where excesses are given by (33). Hyperbolic geometry implies negative θ excess

$$\theta_1+\theta_2+\theta_3 < \pi \quad \Rightarrow \quad 0 < \beta_i, \qquad i = 1, 2, 3. \tag{44}$$

Other choices of signs ϵ_i are analogues of crossing transformation of the hyperbolic triangle.

For arbitrary signs ϵ_i define the Lagrangian density and the action by

$$L(k;k') = -\epsilon_1\epsilon_2\epsilon_3 G(k;k'), \qquad \mathscr{A} = \sum_{n\in\mathbb{Z}^3} L(k_{i,n};k_{i,n+e_i}). \tag{45}$$

The criterion for a correct sign of Lagrangian density is the positive sign near δ^2 in the free-field approximation (30),

$$L(k;k') = (\text{positive coeff.}) \times \delta^2 - V(x) - V_0, \tag{46}$$

so that δ^2 stands for a square of velocity and $V(x) + V_0$ stands for a potential. In this regime, the quadratic form $V(x)$ is positively defined. On the homogeneous solution (ground state)

$$L(k;k) = -V_0 = \int_0^{\beta_4} \log(2\sinh x)\, \mathrm{d}x - \sum_{i=1}^{3} \int_0^{\beta_i} \log(2\sinh x)\, \mathrm{d}x > 0. \tag{47}$$

The global quadratic form has the saddle structure; a general solution of the linearized equations of motion is plane waves with a certain dispersion relation. As is clear for free theory, the value of the whole action on any plane wave solution of the equations of motion in finite volume coincides with its value on the vacuum solution,

$$\mathcal{A} = -N^3 V_0. \tag{48}$$

In the following sections, we give a general solitonic solution of the field-theoretical equations of motion which can be regarded as excitations over the ground state. Dispersion relation for solitons is the same as the dispersion relation for linearized theory. Since for 3D periodical boundary conditions the value of whole action at the equilibrium point is an universal invariant, it does not depend on amplitudes of solitons and therefore it equals the value of the whole action for ground state.

3.3. Regime 3

The unitary parameters u_i are given by

$$u_1 = \mathrm{e}^{\mathrm{i}\epsilon_1\phi_1}, \qquad u_2 = -\mathrm{e}^{\mathrm{i}\epsilon_2\phi_2}, \qquad u_3 = \mathrm{e}^{\mathrm{i}\epsilon_3\phi_3} \tag{49}$$

where as usual $\phi_i > 0$, ϵ_i are the signs and the excesses are defined by (33). This gives for $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$,

$$k_1^2 = \coth^2\frac{\theta_1}{2}, \qquad k_2^2 = \tanh^2\frac{\theta_2}{2}, \qquad k_3^2 = \coth^2\frac{\theta_3}{2}, \tag{50}$$

with cosine theorems

$$\cosh\theta_i = \frac{\cos\phi_i + \cos\phi_j\cos\phi_k}{\sin\phi_j\sin\phi_k}, \qquad \cos\phi_i = \frac{\cosh\theta_j\cosh\theta_k - \cosh\theta_i}{\sinh\theta_j\sinh\theta_k},$$
$$\coth^2\frac{\theta_i}{2} = \frac{\cos\beta_j\cos\beta_k}{\cos\beta_4\cos\beta_i}. \tag{51}$$

This is a hyperbolic triangle formed by an intersection of three planes with time-like normals (and hyperbolic angles θ_i between them) and a one-sheet hyperboloid. Trigonometric sides ϕ_i are defined in the motionless frame of reference for each plane. The time-like normals form a dual triangle on the two-sheets hyperboloid of regime 2. The geometry provides the constraint for β_i:

$$0 < \beta_i < \pi/2, \qquad i = 1, 2, 3, 4, \tag{52}$$

otherwise it would be regime 1. Other choices of signs ϵ_i are analogues of crossing transformation of the hyperbolic triangle.

We define the Lagrangian density for the arbitrary signs ϵ_i by

$$L(k;k') = \mathrm{i}\epsilon_1\epsilon_2\epsilon_3 G(k;k'), \tag{53}$$

where the sign criterion is the same as for regime 2. However, the quadratic potential here is not sign defined. On the homogeneous solution (ground state)

$$L(k;k) = -V_0 = \int_0^{\beta_4} \log(2\cos x)\,\mathrm{d}x - \sum_{i=1}^{3}\int_0^{\beta_i} \log(2\cos x)\,\mathrm{d}x < 0. \tag{54}$$

3.4. Regime 4

Real spectral parameters are defined by

$$u_1 = \mathrm{e}^{-\epsilon_1\phi_1}, \qquad u_2 = -\mathrm{e}^{-\epsilon_2\phi_2}, \qquad u_3 = \mathrm{e}^{-\epsilon_3\phi_3}, \tag{55}$$

where ϕ_i are positive, ϵ_i are again signs. The negative sign of one of the u_i makes the difference with regime 2. This parameterization gives for $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$,

$$k_1^2 = \tanh^2\frac{\theta_1}{2}, \qquad k_2^2 = \coth^2\frac{\theta_2}{2}, \qquad k_3^2 = \tanh^2\frac{\theta_3}{2} \tag{56}$$

where the cosine theorems are

$$\cosh\theta_i = \frac{\cosh\phi_i + \cosh\phi_j\cosh\phi_k}{\sinh\phi_j\sinh\phi_k}, \qquad \cosh\phi_i = \frac{\cosh\theta_i + \cosh\theta_j\cosh\theta_k}{\sinh\theta_j\sinh\theta_k},$$
$$\tanh^2\frac{\theta_i}{2} = \frac{\cosh\beta_j\cosh\beta_k}{\cosh\beta_4\cosh\beta_i}. \tag{57}$$

The hyperbolic excesses are defined by (33). This corresponds to a triangle on the one-sheet hyperboloid with the hyperbolic sides ϕ_i and the hyperbolic dihedral angles θ_i. Such triangle is the section of the one-sheet hyperboloid by planes with space-like normals. Note, two planes with space-like normals and hyperbolic angles between them do not intersect on the two-sheet hyperboloid. In contrast, two planes with the space-like normals and trigonometric angle between them intersect on the two-sheet hyperboloid, this corresponds to regime 2.

Other choices of signs ϵ_i are analogues of crossing transformation of the hyperbolic triangle.

The Lagrangian density and the action are then

$$L(k;k') = \epsilon_1\epsilon_2\epsilon_3 G(k;k'), \qquad \mathscr{A} = \sum_{\boldsymbol{n}\in\mathbb{Z}^3} L(k_{i,\boldsymbol{n}}; k_{i,\boldsymbol{n}+\boldsymbol{e}_i}). \tag{58}$$

In this regime, the quadratic potential is not sign defined either. The ground state gives

$$L(k;k) = -V_0 = \int_0^{\beta_4} \log(2\cosh x)\,\mathrm{d}x - \sum_{j=1}^{3}\int_0^{\beta_j} \log(2\cosh x)\,\mathrm{d}x > 0. \tag{59}$$

4. Solitons

4.1. General soliton solution of (5)

A general (complex) soliton solution of equations (5) is given by the reduction of a general algebraic geometry solution corresponding to the reduction of the genus g curve to a sphere with punches. The resulting expressions are the following [17]. For the number of solitons $g \geqslant 0$ let

$$\{X_j, Y_j\}_{j=1,\dots,g}, \qquad \boldsymbol{f} = \{f_j\}_{j=1,\dots,g} \tag{60}$$

be a set of $3g$ complex values. For the given $\boldsymbol{f}$ and $\{X_j, Y_j\}$ let

$$F_j = f_j \prod_{k\neq j} \frac{X_j - X_k}{Y_j - X_k}. \tag{61}$$

Next we define

$$\Theta(\boldsymbol{f}) = \frac{\det\left|X_j^{k-1} + F_j Y_j^{k-1}\right|_{j,k=1,\dots,g}}{\prod_{i>j}(X_i - X_j)}, \tag{62}$$

where in the numerator there is the determinant of the $g \times g$ matrix with matrix indices j, k. In this expression, f_j is the amplitude of the jth soliton, if one of $f_j = 0$ then (62) simply gives the $g - 1$ soliton expression. For instance,

$$\begin{aligned}
g = 0 \quad &\Rightarrow \quad \Theta = 1,\\
g = 1 \quad &\Rightarrow \quad \Theta(f_1) = 1 + f_1,\\
g = 2 \quad &\Rightarrow \quad \Theta(f_1, f_2) = 1 + f_1 + f_2 + f_1 f_2 \frac{(X_1 - X_2)(Y_1 - Y_2)}{(X_1 - Y_2)(Y_1 - X_2)},
\end{aligned} \tag{63}$$

etc. In general, at the first order of f

$$\Theta(\boldsymbol{f}) = 1 + \sum_{j=1}^{g} f_j + \text{higher terms}, \tag{64}$$

which corresponds to the free-field (linear) approximation. Further, let

$$\omega_{k,j} = \omega_k(X_j, Y_j) = \frac{(Y_j - P_k)(X_j - Q_k)}{(X_j - P_k)(Y_j - Q_k)}, \qquad j = 1, \dots, g, \qquad k = 1, 2, 3 \tag{65}$$

and

$$\boldsymbol{f}(\boldsymbol{n}) = \{f_j(\boldsymbol{n})\}_{j=1,\dots,g}, \qquad f_j(\boldsymbol{n}) = f_j \omega_{1,j}^{n_1} \omega_{2,j}^{-n_2} \omega_{3,j}^{n_3}. \tag{66}$$

Also, for brevity, let

$$\Theta_{\boldsymbol{n}} = \Theta(\boldsymbol{f}(\boldsymbol{n})). \tag{67}$$

The general soliton solution of (5) is then given by [13]

$$\begin{aligned}
k_{1,\boldsymbol{n}}^2 &= \frac{E(Q_2, Q_3)E(P_2, P_3)}{E(Q_2, P_3)E(P_2, Q_3)} \frac{\Theta_{\boldsymbol{n}} \Theta_{\boldsymbol{n}-\boldsymbol{e}_2+\boldsymbol{e}_3}}{\Theta_{\boldsymbol{n}-\boldsymbol{e}_2} \Theta_{\boldsymbol{n}+\boldsymbol{e}_3}},\\
k_{2,\boldsymbol{n}}^2 &= \frac{E(Q_1, Q_3)E(P_1, P_3)}{E(Q_1, P_3)E(P_1, Q_3)} \frac{\Theta_{\boldsymbol{n}-\boldsymbol{e}_2} \Theta_{\boldsymbol{n}+\boldsymbol{e}_1-\boldsymbol{e}_2+\boldsymbol{e}_3}}{\Theta_{\boldsymbol{n}+\boldsymbol{e}_1-\boldsymbol{e}_2} \Theta_{\boldsymbol{n}-\boldsymbol{e}_2+\boldsymbol{e}_3}},\\
k_{3,\boldsymbol{n}}^2 &= \frac{E(Q_1, Q_2)E(P_1, P_2)}{E(Q_1, P_2)E(P_1, Q_2)} \frac{\Theta_{\boldsymbol{n}} \Theta_{\boldsymbol{n}+\boldsymbol{e}_1-\boldsymbol{e}_2}}{\Theta_{\boldsymbol{n}+\boldsymbol{e}_1} \Theta_{\boldsymbol{n}-\boldsymbol{e}_2}},
\end{aligned} \tag{68}$$

where

$$E(Q, P) = \frac{Q - P}{\sqrt{dQdP}} \tag{69}$$

is the prime form on a compact complex plane. Spectral parameters in this parameterization are given by

$$
\begin{aligned}
u_1u_2u_3 &= -\frac{E(Q_1,Q_2)E(P_1,Q_3)E(P_2,P_3)}{E(P_1,P_2)E(Q_1,P_3)E(Q_2,Q_3)},\\
\frac{u_1}{u_2u_3} &= -\frac{E(Q_1,P_2)E(P_1,P_3)E(Q_2,Q_3)}{E(P_1,Q_2)E(Q_1,Q_3)E(P_2,P_3)},\\
\frac{u_2}{u_1u_2} &= -\frac{E(P_1,Q_2)E(Q_1,P_3)E(P_2,Q_3)}{E(Q_1,P_2)E(P_1,Q_3)E(Q_2,P_3)},\\
\frac{u_3}{u_1u_2} &= -\frac{E(P_1,P_2)E(Q_1,Q_3)E(Q_2,P_3)}{E(Q_1,Q_2)E(P_1,P_3)E(P_2,Q_3)}.
\end{aligned}
\tag{70}
$$

4.2. *Identification of parameterizations*

The homogeneous solution of (5) corresponds to $g = 0$ when all $\Theta_n = 1$. Expressions (68) are equivalent to parameterizations of homogeneous k_j^2 in terms of spherical and hyperbolic triangles. Let us demonstrate this statement in more detail.

In regime 1 of the Euclidean spherical trigonometry, consider the planes defined by their unit normal vectors $\vec{n}_i$ in some auxiliary frame of reference,

$$\vec{n}_i = (\sin\vartheta_i\cos\varphi_i, \sin\vartheta_i\sin\varphi_i, \cos\vartheta_i). \tag{71}$$

The dihedral angle between two planes equals the angle between normals,

$$\theta_3 = \widehat{\vec{n}_1\vec{n}_2}, \qquad \theta_2 = \pi - \widehat{\vec{n}_1\vec{n}_3}, \qquad \theta_1 = \widehat{\vec{n}_2\vec{n}_3}, \tag{72}$$

where θ_i are the *inner* dihedral angles of the spherical triangle. Cosine theorems give

$$\cos\theta_1 = (\vec{n}_2,\vec{n}_3) = \cos\vartheta_2\cos\vartheta_3 + \sin\vartheta_2\sin\vartheta_3\cos(\varphi_2-\varphi_3) \quad \text{etc.} \tag{73}$$

For all points, $\vec{n}_i$ on the sphere define their stereographic projections to a complex plane:

$$Q_i = \tan\frac{\vartheta_i}{2}\,\mathrm{e}^{\mathrm{i}\varphi_i}, \qquad P_i = -\cot\frac{\vartheta_i}{2}\,\mathrm{e}^{\mathrm{i}\varphi_i}. \tag{74}$$

The cosine theorem can then be rewritten as

$$-\tan^2\frac{\theta_1}{2} = \frac{\cos\theta_1 - 1}{\cos\theta_1 + 1} = \frac{(Q_2-Q_3)(P_2-P_3)}{(Q_2-P_3)(P_2-Q_3)}, \tag{75}$$

which makes exact correspondence between (35) and (68) for $\Theta_n = 1$.

This can be done similarly for all other regimes. In the Minkowski metric $\boldsymbol{g} = \mathrm{diag}(1,-1,-1)$, a time-like unit vector is parameterized by

$$
\begin{aligned}
&\vec{n}_i = (\cosh\vartheta_i, \sinh\vartheta_i\cos\varphi_i, \sinh\vartheta_i\sin\varphi_i) \quad \text{so that}\\
&\cosh\theta_1 = (\vec{n}_2,\vec{n}_3) = \cosh\vartheta_1\cosh\vartheta_2 - \sinh\vartheta_1\sinh\vartheta_2\cos(\varphi_1-\varphi_2),
\end{aligned}
\tag{76}
$$

which gives for regime 3

$$Q_i = \tanh\frac{\vartheta_i}{2}\,\mathrm{e}^{\mathrm{i}\varphi_i}, \qquad P_i = \coth\frac{\vartheta_i}{2}\,\mathrm{e}^{\mathrm{i}\varphi_i}. \tag{77}$$

The proper parameterization of space-like unit vectors is the following:

$$
\begin{aligned}
&\vec{n}_i = (\tan\vartheta_i, \sec\vartheta_i\cos\varphi_i, \sec\vartheta_i\sin\varphi_i), \qquad \text{so that}\\
&\cos\theta_1 \text{or} \cosh\theta_1 = -(\vec{n}_2,\vec{n}_3) = \frac{\cos(\varphi_2-\varphi_3) - \sin\vartheta_1\sin\vartheta_2}{\cos\vartheta_2\cos\vartheta_3},
\end{aligned}
\tag{78}
$$

which gives for regimes 2,4

$$Q_i = \mathrm{e}^{\mathrm{i}(\varphi_i+\vartheta_i)}, \qquad P_i = -\mathrm{e}^{\mathrm{i}(\varphi_i-\vartheta_i)}. \tag{79}$$

Thus, in terms of complex parameters P_i, Q_i, regimes are classified as follows:

$$\begin{array}{ll} \text{regime 1:} & P_i Q_i^* = -1, \\ \text{regime 3:} & P_i Q_i^* = 1, \\ \text{regimes 2,4:} & |P_i| = |Q_i| = 1, \end{array} \tag{80}$$

where * stands for complex conjugation.

4.3. Plane waves and dispersion relation

Relation (66) stands for a plane wave with exponential frequencies $\omega_k(X, Y), k = 1, 2, 3$. Parameterization (65) can be viewed as a general solution of an algebraic equation relating three $\omega_k, k = 1, 2, 3$. This dispersion relation can be obtained by elimination of X, Y from (65); it has the form

$$\sum_{i,j,k=0}^{2} c_{ijk}(u_1, u_2, u_3)\omega_1^i\omega_2^j\omega_3^k = 0, \tag{81}$$

where $c_{i,j,k}(u_1, u_2, u_3)$ are simple but lengthy rational coefficients. Note that the free-field approximation provides the same dispersion relation.

3D periodical boundary conditions in a rather big volume require unitary ω_k. Parameterization (65) and definition of regimes (80) provide immediately the following:

- In regime 1 it is impossible[2] to make all three $\omega_k(X, Y)$ unitary. Thus, the homogeneous solution is indeed the absolute minimum of the energy functional (38). For the open boundary conditions, the solitons of regime 1 break the signature condition $k_{i,n}^2 < 0$.
- In regimes 2,4, when P_k, Q_k are unitary, all $\omega_k(X, Y)$ are unitary if $X^*Y = 1$.
- In regime 3, when $P_k Q_k^* = 1$, all $\omega_k(X, Y)$ are unitary if $|X| = |Y| = 1$.

In all field-theoretical regimes the reality condition $\Theta_n^* = \Theta_n$ is satisfied for soliton–antisoliton pairs with conjugated amplitudes.

The dispersion relation for ω_k near unity,

$$\omega_1 = \mathrm{e}^{\mathrm{i}p_1}, \qquad \omega_2 = \mathrm{e}^{-\mathrm{i}p_2}, \qquad \omega_3 = \mathrm{e}^{\mathrm{i}p_3}, \tag{82}$$

where momenta p_i are small, reads

$$p_1^2 p_2^2 \left(u_3 - u_3^{-1}\right)^2 + p_1 p_2 p_3^2 \left(u_1 - u_1^{-1}\right)\left(u_2 - u_2^{-1}\right)\left(u_3 + u_3^{-1}\right) + \text{cyclic permutations} = 0. \tag{83}$$

Due to the homogeneouty, this relation describes a cone-type surface in momentum space. In the symmetric cases $u_1 = u_2 = u_3$ or $u_1 = -u_2 = u_3$, which corresponds to $\phi_1 = \phi_2 = \phi_3 = \phi > 0$, relation (83) becomes

$$p_1^2 p_2^2 + p_1^2 p_3^2 + p_2^2 p_3^2 + 2C p_1 p_2 p_3 (p_1 + p_2 + p_3) = 0, \tag{84}$$

where

$$\begin{array}{lll} \text{regime 1:} & C = \cos\phi, & -1/2 < C < 1; \\ \text{regime 2:} & C = \cosh\phi, & 1 < C; \\ \text{regime 3:} & C = -\cos\phi, & -1 < C < -1/2; \\ \text{regime 4:} & C = -\cosh\phi, & C < -1. \end{array} \tag{85}$$

[2] The unitarity condition in regime 1 demands $X^*X = Y^*Y = -1$.

Our four regimes cover the real axis, $C \in \mathbb{R}\backslash\{1, -1, -1/2\}$. We then define an energy E and space-like momenta π_i (we do not care about scales of energy and momenta) by

$$p_i = E + \pi_i, \qquad \pi_1 + \pi_2 + \pi_3 = 0. \tag{86}$$

Let

$$\pi^2 = \frac{1}{2}\left(\pi_1^2 + \pi_2^2 + \pi_3^2\right), \qquad \gamma = \frac{\pi_1\pi_2\pi_3}{\pi^3}. \tag{87}$$

If $C < -1/2$ or $1 < C$, equation (84) defines an anisotropic cone-type surface

$$E = \alpha(\gamma)\pi \tag{88}$$

where $\alpha(\gamma)$ is a real solution of

$$\left(C + \tfrac{1}{2}\right)\alpha^4 - C\alpha^2 + (C-1)\gamma\alpha + \tfrac{1}{6} = 0. \tag{89}$$

Anisotropy parameter γ is bounded,

$$-\gamma_0 \leqslant \gamma \leqslant \gamma_0, \qquad \gamma_0 = \sqrt{\frac{4}{27}}. \tag{90}$$

Critical values $\gamma = \pm\gamma_0$ correspond to three selected directions in the momentum space when

$$p_1 = p_2 = 0 \qquad \text{or} \quad p_1 = p_3 = 0 \qquad \text{or} \quad p_2 = p_3 = 0. \tag{91}$$

When $C < -1/2$ (regimes 3,4), equation (89) has one positive and one negative solution which gives a rather anisotropic 'cone' with

$$\frac{1}{\sqrt{3}}\left(\sqrt{\frac{2C-2}{2C+1}} - 1\right) \leqslant \alpha_+(\gamma) \leqslant \frac{1}{\sqrt{3}}\left(\sqrt{\frac{2C-2}{2C+1}} + 1\right), \tag{92}$$

where $\alpha_+(\gamma)$ are taken as positive (negative solutions for given γ are $\alpha_-(\gamma) = -\alpha_+(-\gamma)$). When $\gamma = \pm\gamma_0$, equation (89) has extra solutions $\alpha = \pm\frac{1}{\sqrt{3}}$; these solutions are isolated and therefore do not belong to a one-parameter family, and they have no relation to ω_i and should be ignored.

When $1 < C$ (regime 2), equation (89) has two positive and two negative solutions which give two imbedded tangent anisotropic 'cones'. The 'cones' are tangent along $\gamma = \pm\gamma_0, \alpha(\pm\gamma_0) = \pm\frac{1}{\sqrt{3}}$; these points are not isolated. The existence of two 'speeds of light' is a surprise.

Regime 2 involves the Lorentz group limit. If all ϕ_i in this regime are small, $\cos\phi_1 \simeq 1$, then the dispersion relation (83) becomes

$$\left(\frac{p_1}{\phi_1}\frac{p_2}{\phi_2} + \frac{p_1}{\phi_1}\frac{p_3}{\phi_3} + \frac{p_2}{\phi_2}\frac{p_3}{\phi_3}\right)^2 = 0, \tag{93}$$

which is equivalent to an isotropic light cone and gives the pure Minkowski metric in the momentum space.

When $-1/2 < C < 1$ (regime 1), equation (89) has no real solutions as expected.

5. Quantum theories

In this section we discuss the relation of classical regimes with quantum models.

We commence with a short reminder of quantum R-matrices. Let $\mathcal{A}$ be the enveloping of the q-oscillator algebra

$$\boldsymbol{k}\boldsymbol{a}^{\pm} = q^{\pm 1}\boldsymbol{a}^{\pm}\boldsymbol{k}, \qquad \boldsymbol{a}^{+}\boldsymbol{a}^{-} = 1 - q^{-1}\boldsymbol{k}^2, \qquad \boldsymbol{a}^{-}\boldsymbol{a}^{+} = 1 - q\boldsymbol{k}^2. \tag{94}$$

equipped by a pair of $\mathbb{C}$-valued parameters λ, μ,

$$\mathcal{A} = (1, \boldsymbol{k}, \boldsymbol{a}^{\pm}, \lambda, \mu). \tag{95}$$

The map $\boldsymbol{R}_{123}$ [1, 2] of tensor cube $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_3$ is defined by (confer with (5))

$$\begin{aligned}
\boldsymbol{R}_{123}\boldsymbol{k}_2\boldsymbol{a}_1^{\pm}\boldsymbol{R}_{123}^{-1} &= u_1^{\pm 1}\left(\boldsymbol{k}_3\boldsymbol{a}_1^{\pm} + u_2^{\mp 1}\boldsymbol{k}_1\boldsymbol{a}_2^{\pm}\boldsymbol{a}_3^{\mp}\right),\\
\boldsymbol{R}_{123}\boldsymbol{a}_2^{\pm}\boldsymbol{R}_{123}^{-1} &= \boldsymbol{a}_1^{\pm}\boldsymbol{a}_3^{\pm} - u_2^{\mp 1}\boldsymbol{k}_1\boldsymbol{k}_3\boldsymbol{a}_2^{\pm},\\
\boldsymbol{R}_{123}\boldsymbol{k}_2\boldsymbol{a}_3^{\pm}\boldsymbol{R}_{123}^{-1} &= u_3^{\pm 1}\left(\boldsymbol{k}_1\boldsymbol{a}_3^{\pm} + u_2^{\mp 1}\boldsymbol{k}_3\boldsymbol{a}_1^{\mp}\boldsymbol{a}_2^{\pm}\right),
\end{aligned} \tag{96}$$

where

$$u_1 = \frac{\lambda_3}{\lambda_2}, \qquad u_2 = -\frac{1}{\lambda_1\mu_3}, \qquad u_3 = \frac{\mu_1}{\mu_2}, \tag{97}$$

satisfies the adjoint tetrahedron equation in $\mathcal{A}^{\otimes 6}$ and the quantum tetrahedron equation in proper $\mathrm{Rep}(\mathcal{A})^{\otimes 6}$ (spectral parameters $\lambda_1, \mu_1, \ldots \lambda_6, \mu_6$ for the tetrahedron equation are free). Equations (96) provide in addition

$$\boldsymbol{R}_{123}\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{R}_{123}^{-1} = \boldsymbol{k}_1\boldsymbol{k}_2, \qquad \boldsymbol{R}_{123}\boldsymbol{k}_2\boldsymbol{k}_3\boldsymbol{R}_{123}^{-1} = \boldsymbol{k}_2\boldsymbol{k}_3. \tag{98}$$

'Constant' matrix $\boldsymbol{r}_{123}$ corresponds to $u_1 = u_2 = u_3 = 1$. In modular representation

$$q = \mathrm{e}^{\mathrm{i}\pi b^2}, \qquad \boldsymbol{k} = -\mathrm{i}\mathrm{e}^{\pi\sigma b}, \qquad b > 0 \qquad \text{and} \qquad \sigma \in \mathbb{R}, \tag{99}$$

the kernel of the constant $\boldsymbol{r}$-matrix is given by [1]

$$\begin{aligned}
&\langle\sigma_1\sigma_2\sigma_3|\boldsymbol{r}|\sigma_1'\sigma_2'\sigma_3'\rangle = \delta_{\sigma_1+\sigma_2,\sigma_1'+\sigma_2'}\delta_{\sigma_2+\sigma_3,\sigma_2'+\sigma_3'}\sqrt{\frac{\varphi(\sigma_1)\varphi(\sigma_2)\varphi(\sigma_3)}{\varphi(\sigma_1')\varphi(\sigma_2')\varphi(\sigma_3')}}\\
&\mathrm{e}^{-\mathrm{i}\pi(\sigma_1\sigma_3-\mathrm{i}\eta(\sigma_1+\sigma_3-\sigma_2'))}\int_{\mathbb{R}}\mathrm{d}u\,\mathrm{e}^{2\pi\mathrm{i}u(\sigma_2'-\mathrm{i}\eta)}\frac{\varphi\left(u+\frac{\sigma_1'+\sigma_3'+\mathrm{i}\eta}{2}\right)\varphi\left(u+\frac{-\sigma_1-\sigma_3+\mathrm{i}\eta}{2}\right)}{\varphi\left(u+\frac{\sigma_1-\sigma_3-\mathrm{i}\eta}{2}\right)\varphi\left(u+\frac{\sigma_3-\sigma_1-\mathrm{i}\eta}{2}\right)}
\end{aligned} \tag{100}$$

where $\varphi(z)$ is the 'non-compact quantum dilogarithm' [7] defined by

$$\varphi(z) = \exp\left(\frac{1}{4}\int_{\mathbb{R}+\mathrm{i}0}\frac{\mathrm{e}^{-2\mathrm{i}zw}}{\sinh(wb)\sinh(w/b)}\frac{\mathrm{d}w}{w}\right), \qquad \frac{\varphi(z-\mathrm{i}b^{\pm 1}/2)}{\varphi(z+\mathrm{i}b^{\pm 1}/2)} = 1 + \mathrm{e}^{2\pi z b^{\pm 1}}. \tag{101}$$

Crossing parameter η in (100) is given by

$$\eta = \frac{b+b^{-1}}{2}. \tag{102}$$

In Fock space (F^+) and anti-Fock space (F^-) representations

$$q = \mathrm{e}^{-\varepsilon}, \quad \boldsymbol{k} = q^{n+1/2}, \quad n = 0, 1, 2, 3\ldots(F^+) \qquad \text{or} \quad n = -1, -2, -3, \ldots(F^-), \tag{103}$$

the matrix elements of the constant $\boldsymbol{r}$-matrix are given by a similar formula [1, 2, 21],

$$\begin{aligned}
&\langle n_1n_2n_3|\boldsymbol{r}|n_1'n_2'n_3'\rangle = \delta_{n_1+n_2,n_1'+n_2'}\delta_{n_2+n_3,n_2'+n_3'}\prod_{i=1}^{3}c_{n_i,n_i'}\\
&q^{n_1n_3+n_2'}\frac{1}{2\pi\mathrm{i}}\oint\frac{\mathrm{d}z}{z^{n_2'+1}}\frac{(-q^{2+n_1'+n_3'}z;q^2)_\infty(-q^{-n_1-n_3}z;q^2)_\infty}{(-q^{+n_1-n_3}z;q^2)_\infty(-q^{-n_1+n_3}z;q^2)_\infty}
\end{aligned} \tag{104}$$

where

$$c_{n,n'} = \sqrt{\frac{(q^2;q^2)_{n'}}{(q^2;q^2)_n}} \qquad \text{if} \quad n = 0, 1, 2, 3\ldots(F^+) \tag{105}$$

or

$$c_{n,n'} = \sqrt{\frac{q^{n'(n'+1)}(q^2;q^2)_{-n-1}}{q^{n(n+1)}(q^2;q^2)_{-n'-1}}} \qquad \text{if} \quad n = -1, -2, -3, -4 \ldots (F^-) \tag{106}$$

The *clockwise* integration loop in (104) circles all poles of the integrand but does not include $z = 0$. Pochhammer's symbols and Euler's quantum dilogarithm are defined by

$$(z;q^2)_n = (1-z)(1-q^2z)\cdots(1-q^{2(n-1)}z), \qquad \frac{(-z/q;q^2)_\infty}{(-qz;q^2)_\infty} = 1 + z/q. \tag{107}$$

Matrix (104) has the block-diagonal structure in

$$F_1^{\epsilon_1} \otimes F_2^{\epsilon_2} \otimes F_3^{\epsilon_3}, \qquad \epsilon_i = \pm, \tag{108}$$

and thus it defines eight different R-matrices. Classical limits of Fock and anti-Fock space representations (103) provide

$$0 < k < 1 \quad \text{for} \quad F^+ \qquad \text{and} \qquad 1 < k \quad \text{for} \quad F^-. \tag{109}$$

Pre-factors and integrands in both (100) and (104) have identical difference properties (leftmost relations in (101) and (107)); the main difference is that there is a non-compact set of poles in the modular integrand and there is a compact set of poles in the Fock space integrand.

The advantage of the special case $u_1 = u_2 = u_3 = 1$ is that the constant $\boldsymbol{r}$ is the symmetric root of unity,

$$\boldsymbol{r}_{123}^2 = 1, \qquad \boldsymbol{r}_{123}^\dagger = s_1 s_2 s_3\, \boldsymbol{r}_{123} (s_1 s_2 s_3)^{-1}, \tag{110}$$

where $s = 1$ for modular representation and Fock representation F^+ and

$$s = (-)^n \qquad \text{for} \quad F^-. \tag{111}$$

Factor s takes into account the anti-unitarity $(\boldsymbol{a}^\pm)^\dagger = -\boldsymbol{a}^\mp$ of anti-Fock representations. For instance, the matrix

$$\boldsymbol{r}'_{123} = (-)^{n_2} \boldsymbol{r}_{123} \qquad \text{in} \quad F_1^+ \otimes F_2^- \otimes F_3^+ \tag{112}$$

is Hermitian.

Spectral parameters in (96) are given by 'external field' factors. All cases below correspond to spectral parameters (32, 42, 49, 55) with positive signs $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$.

5.1. Regime 1

The R-matrix of (96) in modular representation $\boldsymbol{k}^2 < 0$ (99) and spectral parameters of regime 1 are given by

$$\boldsymbol{R}_{123} = \mathrm{e}^{-2\eta\phi_2\sigma_2} \boldsymbol{r}_{123}\, \mathrm{e}^{2\eta\phi_1\sigma_1 + 2\eta\phi_3\sigma_3}. \tag{113}$$

The kernel of (113) is real and in the vicinity of the equilibrium point (35) it is positive with the asymptotic

$$\langle\sigma_1\sigma_2\sigma_3|\boldsymbol{R}_{123}|\sigma_1\sigma_2\sigma_3\rangle \sim \exp\left(-\frac{F_0}{\pi b^2}\right) \qquad \text{as} \quad b \to 0 \qquad \text{and} \qquad -\mathrm{i}\,\mathrm{e}^{\pi b \sigma_i} \to k_i, \tag{114}$$

where free energy F_0 as a function of $\phi_{1..3}$ is given by (39). Presumably, partition function per site for cubic lattice and the R-matrix (113) in physical regime $0 < \beta_j < \pi$ is

$$z = \exp\left(-\frac{4\eta^2}{\pi} F_0\right) \tag{115}$$

for arbitrary $\eta > 0$ (102).

5.2. Regime 2

The R-matrix of (96) in modular representation $\boldsymbol{k}^2 < 0$ (99) and spectral parameters of regime 2 are given by

$$\boldsymbol{R}_{123} = \varrho^{-1}\,\mathrm{e}^{2\mathrm{i}\eta\phi_2\sigma_2}\boldsymbol{r}_{123}\,\mathrm{e}^{-2\mathrm{i}\eta\phi_1\sigma_1-2\mathrm{i}\eta\phi_3\sigma_3}, \qquad (116)$$

where ϱ is a unitary constant multiplier. In the vicinity $-\mathrm{i}\mathrm{e}^{\pi b\sigma_i} \to k_i$ of ground state (35) for regime 2 the kernel of the constant $\boldsymbol{r}$-matrix oscillates. R-matrix (116) is unitary and therefore it is the building block for a Heisenberg evolution operator. However, a spectral equation for the evolution operator is not yet known and we cannot rigorously deduce a relation between spectra of quantum field theory and solitons and dispersion relation (83) of classical field theory.

5.3. Regime 3

A self-consistent quantum field theory for Fock space representations corresponding to spectral parameters (49) and (50) is defined by

$$\boldsymbol{R}_{123} = \varrho^{-1}\,\mathrm{e}^{\mathrm{i}(\pi-\phi_2)n_2}\boldsymbol{r}_{123}\,\mathrm{e}^{\mathrm{i}\phi_1 n_1+\mathrm{i}\phi_3 n_3} \qquad \text{in} \quad F_1^- \otimes F_2^+ \otimes F_3^-. \qquad (117)$$

The constant $\boldsymbol{r}$-matrix in $F_1^- \otimes F_2^+ \otimes F_3^-$ oscillates, operator (117) is the unitary one for unitary constant multiplier ϱ. A spectral equation for the Heisenberg evolution operator is not known either except for a special 1 + 1 dimensional case and small occupation numbers [19]. In the same way as for regime 2 we cannot deduce rigorously relations between quantum spectra and classical dispersion relation. Note however an interesting feature of regime 3: a self-consistency prescribes a correspondence between signatures ϵ_i of spectral parameters and a choice of representation F^+ with $0 < k < 1$ or F^- with $k > 1$. Presumably, this correspondence provides a proper physical interpretation of spectra of evolution operators (see [19] for 1 + 1 dimensional case). A choice of constant ϱ in both regimes 2 and 3 is also a subject of proper physical interpretation.

5.4. Regime 4

Curiously, the field-theoretical regime 4 has no quantum field-theoretical counterpart; it corresponds to divergent statistical mechanics.

The R-matrix for regime 4 (55) is given by

$$\boldsymbol{R}_{123} = \mathrm{e}^{\phi_2 n_2}\,\boldsymbol{r}'_{123}\,\mathrm{e}^{-\phi_1 n_1-\phi_2 n_2} \qquad \text{in} \quad F_1^+ \otimes F_2^- \otimes F_3^+ \qquad (118)$$

where $\boldsymbol{r}'$ is defined by (112) and representation $F_1^+ \otimes F_2^- \otimes F_3^+$ is chosen in accordance with (56). Matrix elements of (118) are strictly positive and diverge as

$$\langle n|\boldsymbol{R}|n\rangle \sim q^{n_1n_2-n_1n_3+n_2n_3+\text{lower terms}} \qquad \text{as} \quad n_1, n_3 \to \infty, \qquad n_2 \to -\infty. \qquad (119)$$

A well-defined statistical mechanical lattice theory should involve a compensation of quadratic exponential asymptotic. It is possible via certain non-linear boundary conditions preserving the integrability and involving extra three temperature-like parameters.

In the quasi-classical limit $q = \mathrm{e}^{-\varepsilon} \to 1$, the diagonal matrix element of $\boldsymbol{r}'$ is given by

$$\langle n|\boldsymbol{r}'|n\rangle \sim \mathrm{e}^{\phi_1 n_1-\phi_2 n_2+\phi_3 n_3}\exp\left(\frac{V_0}{\varepsilon}\right) \qquad \text{as} \quad \varepsilon \to 0, \qquad (120)$$

where finite $k_i = q^{n_i}$ define by (56) and (57) the hyperbolic triangle with dihedral angles θ_i and positive sides ϕ_i, V_0 is then given by (59). For so defined ϕ_i the field factor in (118) compensates the pre-exponent in (120). However, this 'classical equilibrium point' has no relation with a self-consistent quantum model.

5.5. Regimes 2 and 3 as gauge field theories

Well-defined quantum field theories in regimes 2 and 3 involve Bose q-oscillators. However, the algebraic approach to the quantum tetrahedron equations allows one to introduce Fermi oscillators in addition to Bose ones [21]. All fermionic R-matrices are even, and they involve two fermions and one boson. Both fermionic and bosonic R-matrices are building blocks of the Heisenberg evolution operator. One can straightforwardly consider an evolution of simple test states with small total occupation numbers (in the Fock space representation for bosons). The evolution produces a set of Feynmann diagrams on a constant time discrete surface (kagome lattice). In addition to simple propagation, fermionic R-matrices are responsible for emissions of bosons, the decay of a boson into a fermion pair and the annihilation of a fermion pair into a boson. Thus, the interpretation of quantum field theories as gauge field theories, where the bosons are gauge fields and fermions are matter field, is quite natural. Presumably, a proper choice of spectral parameters provides also a gap between bosonic ground state and fermionic ground state; thus the spectral parameters are also responsible for fermionic mass.

6. Discussion: algebraic curves of higher *genera*

Formulas (68) for the soliton solution from the previous sections formally coincide with those for a general complex algebraic geometry (finite gap) solution: P_i, Q_i are divisors on a genus g algebraic curve Γ_g, E is a prime form on it, $\boldsymbol{f}$ is related to a point on $\mathrm{Jac}(\Gamma_g)$ and $\Theta_{\boldsymbol{n}}$ is a theta-function:

$$\Theta_{\boldsymbol{n}} = \Theta(I(\boldsymbol{n})), \qquad I(\boldsymbol{n}) = \boldsymbol{z} + n_1 \int_{Q_1}^{P_1} \boldsymbol{\omega} - n_2 \int_{Q_2}^{P_2} \boldsymbol{\omega} + n_3 \int_{Q_3}^{P_3} \boldsymbol{\omega} \in \mathrm{Jac}(\Gamma_g), \tag{121}$$

where $\boldsymbol{\omega}$ is a vector of Abel's holomorphic differentials and $\boldsymbol{z}$ is an arbitrary point on the Jacobian. Any three-terms relation (5) is just the Fay identity [16]. The expression for $I(\boldsymbol{n})$ in (121) corresponds to a special case of homogeneous divisors. Divisors P_i, Q_i are not free, they are divisors of three meromorphic functions,

$$N \int_{Q_i}^{P_i} \boldsymbol{\omega} = 0 \qquad \mathrm{mod}(\pi, \pi\Omega), \qquad i = 1, 2, 3, \tag{122}$$

where N is a size of cubic lattice, and Ω is a period matrix; equations (122) provide the periodical boundary conditions.

The soliton solution is not just a straightforward trigonometric limit of algebraic geometry one since conditions (122) are relaxed, P_i and Q_i in a general complex soliton solution are free.

For the discrete time evolution system, the initial data of the Cauchy problem define uniquely the algebraic curve [13] and thus selects the cases of soliton or finite gap dynamics.

For general finite gap dynamics the spectral parameters u_i in (5) are not uniquely defined. The reason is that equations of motion (5) have gauge invariance. The gauge transformation

$$a_{j,\boldsymbol{n}}^{\pm} \to \xi_{j,\boldsymbol{n}}^{\pm 1} a_{j,\boldsymbol{n}}^{\pm} \tag{123}$$

such that

$$\xi_{2,\boldsymbol{n}+\boldsymbol{e}_2} = \xi_{1,\boldsymbol{n}} \xi_{3,\boldsymbol{n}} \tag{124}$$

is equivalent to the transformation of spectral parameters

$$u_1 \to u_1 \frac{\xi_{1,n}}{\xi_{1,n+e_1}}, \qquad u_2 \to u_2 \frac{\xi_{2,n+e_2}}{\xi_{2,n}}, \qquad u_3 \to u_3 \frac{\xi_{3,n}}{\xi_{3,n+e_3}}. \tag{125}$$

The existence of a homogeneous solution of equations of motion (5) fixes the gauge group element and thus provides the definition of u_j. Otherwise, parameters u_j are irrelevant.

Based on the principles of quantum-classical correspondence, one can conclude that the canonical quantization of q-oscillators (94) and the choice of the Hilbert space as the product of local irreducible representations of q-oscillators corresponds to the choice of soliton sector on classical equations of motion. In particular, the condition of polynomial structure of Q-operators for a nested Bethe ansatz for Fock space representations literally corresponds to factorization of a spectral curve. The finite gap sector must thus correspond to *another* quantization scheme—a finite-gap quantization.

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